Int. J. Solids Structures Vol. 19. No. 6, pp. 509–518, 1983
Printed in Great Britain.

0020-7683/83/060509-10$03.00/0

ON TORSIONAL BUCKLING OF THIN WALLED I COLUMNS WITH VARIABLE CROSS-SECTION

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(*Received* 1 *June* 1981; *in revised form* 18 *October* 1982)

Abstract—The problem of extreme critical conservative loads of torsional buckling for axially compressed thin walled I columns with variable, within given limits, bisymmetric cross-section, is considered. Basing on the Pontryagin's maximum principle, it has been shown that the critical load of a column with variable cross-section may exceed the bounds defined by critical loads for columns with constant extreme cross-section. On the other hand the extreme loads for flexural buckling are the critical ones for columns with extreme constant cross-section. In the numerical example, enclosed, the extreme critical loads for simply supported I column with variable width of flanges and corresponding optimal shapes of flanges are the object of analysis. Moreover, it has been shown that these bifurcation points are symmetric and stable.

NOTATION

A	cross-sectional area
b	width of flanges of the I column
B	bimoment
d	design variable
d_f, d_w	thickness of flanges and web of the I column, respectively
E	Young's modulus
G	shear modulus
h	height of web of the I column
H	Hamiltonian
I	second moment of inertia
I_d	St. Venant's torsional constant
I_0	polar moment of inertia
I_ω	warping torsional constant
k	constant
l	length of the I column
M	torsional moment
M_b	bending moment
P	axial end loads
r	distance from a point of cross-section to the cross-section centroid
u, v, w	displacements of a point of cross-section in the x-, y- and z-axes direction, respectively
V, V_i, V_e	total potential energy, total strain energy, and potential energy of the applied loads, respectively
x, y, z	Cartesian coordinates through the centroid of cross-section
ϵ_z	strain the z-axis direction
Θ	rotational displacement of cross-section
λ	coefficient ($=\sqrt{((P\frac{I_0}{A} - GI_d)/EI_\omega)}$
ϑ	$=\Theta'$
$\psi_\Theta, \psi_\varphi, \psi_B, \psi_M, \bar{\Theta}, \bar{\varphi}, \bar{B}, \bar{M}$	adjoint variables
ω	sectorial area
$(\ldots)'$	first derivative with respect to z ($= d(\ldots)/(dz)$)
$\frac{\partial(\ldots)}{\partial b}$	first partial derivative with respect to b
$\delta(\ldots)$	first variation of $(\ldots)$

Subscripts

cr	critical value
max	maximum value
min	minimum value
0	initial value at $z = 0$
opt	optimal value

1. INTRODUCTION

It is well known [1] that for thin walled columns with bisymmetric open cross-sections the torsional buckling under conservative loads is independent of the flexural buckling.

The results of the buckling analysis of the I column with parabolic variable width of flanges[2] where the critical load of torsional buckling is higher than the same one for column of constant maximum cross-section, indicate the difference between these two kinds of buckling. In Refs. [3,4] the results have not only been confirmed, by means of applying a different method, but there has also been presented the case of step-variable width of the I column flanges, in which the critical load is smaller than the same load for constant minimum cross-section. It is worthwhile moticing that the above mentioned property of the critical load for the flexural buckling does not hold in compliance with one of the theorems given by Milner and Horne[5] and hereafter it will be proved in a different manner.

The main purpose of the paper is to examine in a more rigorous way the property of the critical load of the torsional buckling. It is assumed that only one dimension of the cross-section, apart from the height of the web, may be variable along the axis of the I column. The constant minimum and maximum constraints are imposed on the design variable. Using Pontryagin's maximum principle[7] the optimality condition for the design variable is settled, and thus it is possible to prove the property under consideration. In the numerical examples the width of flanges is assumed to be the design variable and the extreme critical loads as well as the corresponding optimal shapes of flanges are found. Moreover, taking advantage of the analysis carried out by Szymczak[6], the initial postbuckling behavior of these optimal I columns after torsional buckling is discussed. The considerations are based upon the classical assumptions of the thin-walled beam theory[1]:

(1) A cross-section of the column is not deformable.
(2) The shear deformation is negligible.
(3) The strains are small and elastic.

2. FUNDAMENTAL DIFFERENTIAL EQUATION

Let us consider a thin-walled I column with variable bisymmetric cross-section, as shown in Fig. 1, and let the origin of the orthogonal coordinate axes be placed at the centroid of the cross-section. As mentioned before, the height of the web is constant, but other dimensions may be variable along the z-axis of the column. Let us assume moreover that in case when the column buckles due to conservative axial end loads P_{cr}, the cros -section $z = \text{constant}$ moves along z-axis and rotates around the same axis as rigid contour. The displacements u, v of a cross-section point x, y in the x- and y-axes direction are given by

$$u = y\Theta, \quad v = x\Theta. \tag{1}$$

The non-zero component of Green's strain tensor in the z-axis direction ϵ_z can be derived from the well-known nonlinear relationship[6], in which the assumption of the small strains is taken into account

$$\epsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right]. \tag{2}$$

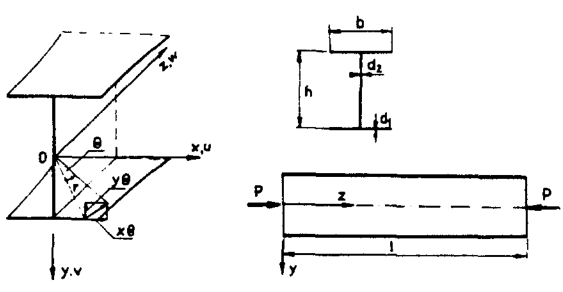

Fig. 1. State of displacements and I column geometry.

According to linear theory of thin walled beam (see, e.g. [1]) the first term in eqn (2) may be written as

$$\frac{\partial w}{\partial z} = \bar{w}' - \omega\Theta'', \tag{3}$$

where the axial displacement of the whole cross-section is denoted by $\bar{w}$. Substituting eqns (1) and (3) into eqn (2) we obtain

$$\epsilon_z = \bar{w}' - \omega\Theta'' + \frac{1}{2}r^2\Theta'^2, \tag{4}$$

where $r^2 = x^2 + y^2$. It is worthwhile noticing that a more detailed description of the strain state has been presented in Ref. [6]. Now, it is possible to determine the total potential energy V of the column. It consists of the total strain energy V_i and the potential energy of the compressive axial end loads V_e

$$V = V_i + V_e. \tag{5}$$

The total strain energy of the column of length l can be written as the sum of the internal strain energy of normal stresses and the internal energy corresponding to the free torsion

$$V_i = \frac{1}{2}E\int_0^l \int_A \epsilon_z^2 \, dA \, dz + \frac{1}{2}G\int_0^l I_d\Theta'^2 \, dz. \tag{6}$$

Furthermore substituting eqn (4) into eqn (6) and integrating the first term over the whole cross-section area we have

$$V_i = \frac{1}{2}\int_0^l \left(EA\bar{w}'^2 + EI_\omega\Theta''^2 + EI_0\Theta'^2\bar{w}' + GI_d\Theta'^2\right) dz, \tag{7}$$

where I_ω and I_0 denote the warping torsional constant and the polar moment of inertia, respectively. The axial displacement of the cross-section after torsional buckling may be expressed as sum of the displacement $\bar{w}_0(z)$ due to axial end loads and the longitudinal displacement $\bar{w}_1(z)$ caused by the twist of the column

$$\bar{w}(z) = \bar{w}_0(z) + \bar{w}_1(z) = -\frac{P_{cr}z}{EA} + \bar{w}_1(z). \tag{8}$$

Using eqns (5), (7) and (8) the total potential energy of the column can be written as

$$V = \frac{1}{2}\int_0^l \left[EA\left(\bar{w}_1' - \frac{P_{cr}}{EA}\right)^2 + EI_\omega\Theta''^2 + EI_0\left(\bar{w}_1' - \frac{P_{cr}}{EA}\right)\Theta'^2 + GI_d\Theta'^2 + P_{cr}\left(\bar{w}_1' - \frac{P_{cr}}{EA}\right)\right] dz, \tag{9}$$

where the potential energy of axial end loads is expressed by the last term in square brackets. The first Euler condition of stationary potential energy leads to the relation

$$\bar{w}_1' = -\frac{I_0}{2A}\Theta'^2. \tag{10}$$

The substitution of eqn (10) into eqn (9) enables the total potential energy to be expressed only in the rotational displacement Θ

$$V = \frac{1}{2}\int_0^l \left(EI_\omega\Theta''^2 + GI_d\Theta'^2 - P_{cr}\frac{I_0}{A}\Theta'^2\right) dz, \tag{11}$$

where the constant terms are neglected. From the Euler condition of stationary potential energy of the column (11) the fundamental differential equation is derived

$$(EI_\omega \Theta'')'' + \left[\left(P_{cr}\frac{I_0}{A} - GI_d\right)\Theta'\right]' = 0, \tag{12}$$

together with the boundary conditions

$$\left[\left(GI_d - P_{cr}\frac{I_0}{A}\right)\Theta' - (EI_\omega \Theta'')'\right]\delta\Theta\Big|_0^l = 0, \quad EI_\omega \Theta''\delta\Theta'\Big|_0^l = 0. \tag{13}$$

It should be noted that the fundamental equation (12) has also been derived in a different manner in Ref. [2], but with respect to somewhat controversial subject of the paper it seems necessary to present the independent derivation of the equation.

3. OPTIMIZATION PROBLEM

Let us denote the design variable by d(z) and impose on it the geometrical constraints constant along the column axis

$$d_{\min} \leq d(z) \leq d_{\max}. \tag{14}$$

The optimization problem can be stated as follows: among all the admissible design variables $d(z)$ which satisfy the geometrical constraints (14), the optimal ones corresponding to the maximum and minimum value of the critical load P_{cr} are sought.

The solution of the problem is obtained by means of the Pontryagin's maximum principle [7]. Equation (12) can be replaced by the following system of first order differential equations

$$\begin{Bmatrix} \Theta \\ \varphi \\ B \\ M \end{Bmatrix}' = \begin{Bmatrix} \varphi \\ -\dfrac{B}{EI_\omega} \\ M + \left(P_{cr}\dfrac{I_0}{A} - GI_d\right)\varphi \\ 0 \end{Bmatrix}. \tag{15}$$

Here some new variables φ-first derivative of Θ, B-bimoment and M-torsional moment are introduced. The Hamiltonian can be constructed by using the standard procedure

$$H = \psi_\Theta \varphi - \psi_\varphi \frac{B}{EI_\omega} + \psi_B\left[M + \left(P_{cr}\frac{I_0}{A} - GI_d\right)\varphi\right]. \tag{16}$$

The adjoint variables ψ_Θ, ψ_φ and ψ_B can be found from the adjoint system of equations

$$\begin{Bmatrix} \psi_\Theta \\ \psi_\varphi \\ \psi_B \\ \psi_M \end{Bmatrix}' = -\begin{Bmatrix} \dfrac{\partial H}{\partial \Theta} \\ \dfrac{\partial H}{\partial \varphi} \\ \dfrac{\partial H}{\partial B} \\ \dfrac{\partial H}{\partial M} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\psi_\Theta - \left(P_{cr}\dfrac{I_0}{A} - GI_d\right)\psi_B, \\ \dfrac{\psi_\varphi}{EI_\omega} \\ -\psi_B \end{Bmatrix} \tag{17}$$

where ψ_M is also an adjoint variable. Introducing an exchange of the adjoint variables

$$\psi_\Theta = \bar{M}, \quad \psi_\varphi = -\bar{B}, \quad \psi_B = \bar{\varphi}, \quad \psi_M = -\bar{\Theta}, \tag{18}$$

to the adjoint system of eqns (17), it is easy to show that we obtain a system of differential equations analogous to the basic system (15) for the new adjoint variables $\bar{\Theta}$, $\bar{\varphi}$, $\bar{B}$, $\bar{M}$. Similarly, one can see that this analogy may be extended to linear boundary conditions. It is worth noticing that this property holds with regard to selfadjointness of the differential operator (12). Basing on this analogy the following relationships can be written

$$\bar{\Theta} = k\Theta, \quad \bar{\varphi} = k\varphi, \quad \bar{B} = kB, \quad \bar{M} = kM, \tag{19}$$

where k is a constant. Using the eqn (19) the Hamiltonian (16) can be rewritten in the form

$$H = k\left\{\varphi^2 + \frac{B^2}{EI_\omega} + \left[M + \left(P_{cr}\frac{I_0}{A} - GI_d\right)\varphi\right]\varphi\right\}. \tag{20}$$

The Pontryagin's maximum principle[7] states that for optimal design variables $d_{opt}(z)$ corresponding to the extreme critical loads, the Hamiltonian reaches its maximum. From this principle, neglecting the terms independent of the design variable in equation (20), one can obtain an optimality condition to determine the optimal design variable $d_{opt}(z)$

$$\max_d H = \max_d k\left[\frac{B^2}{EI_\omega} + \left(P_{cr}\frac{I_0}{A} - GI_d\right)\varphi^2\right] \rightarrow d_{opt}(z), \tag{21}$$

where subscript d denotes the maximum with respect to design variable $d(z)$. The same problem may be stated with reference to the flexural buckling of the column. In this case the differential equation has the form

$$(EIv'')'' + P_{cr}v'' = 0, \tag{22}$$

where I is the second moment of inertia, and v denotes a deflection in the buckling plane. Similarly to the above procedure we can obtain the optimality condition valid in this case

$$\max_d H = \max_d \frac{M_b^2}{EI} \rightarrow d_{opt}(z), \tag{23}$$

where M_b is the bending moment. Taking into account that the second moment of inertia I is a monotonously increasing function of the design variable d, the optimality condition allows us to draw a general conclusion on the optimal design variable, namely, the maximum critical load is obtained if $d(z) = d_{max}$ and the minimum load, if $d(z) = d_{min}$. However, in the torsional buckling problem the same conclusion is generally not valid and for the optimal design variables which satisfy the geometrical constraints (14) the critical loads may exceed the bounds defined by the critical loads for the columns with constant extreme design variables.

4. METHOD OF SOLUTION

In further considerations, without loss of generality, the width of flanges $b(z)$ is assumed to be the design variable. The geometrical constraints imposed on the width of flanges can be written as follows

$$b_{min} \leq b(z) \leq b_{max}. \tag{24}$$

Substituting the following geometrical relations

$$I_\omega = \frac{h^2 d_f b^3}{24}, \quad \frac{I_0}{A} = \frac{d_w h^3 + 6d_f bh^2 + 2d_f b^3}{12(d_w h + 2d_f b)}, \tag{25}$$

$$I_d = \frac{1}{3}(2d_f^3 b + d_w^3 h),$$

in which d_f and d_w are thicknesses of flanges and web, respectively, into the optimality condition (21) we obtain the condition for determination of optimal width $b_{opt}(z)$. Unfortunately, this condition is very complicated and it is impossible to find the solution in an analytical form. Therefore, the extreme critical loads and their corresponding shapes of flanges are determined in an iterative way.

The iterative procedure looks as follows:

Step 1. Choose an initial shape of the flanges $b(z)$ which satisfies the geometrical constraints (24).

Step 2. Having divided the column into finite elements with average constant cross-section compute the critical load P_{cr} and the corresponding eigenvector by means of the stiffness matrix and the geometrical matrix derived in Ref. [3].

Step 2. Compute the values of the variables $B(z)$ and $\varphi(z)$ in the middle of each element by applying eqn (15) in which the differentiation is replaced by finite difference scheme.

Step 4. From the optimality condition (21), using the Newton algorithm compute the new width of flanges in each element so that the geometrical constraints (24) would be satisfied.

Step 5. If the change of the critical load is sufficiently small terminate. Otherwise, return to Step 2 with the new width of flanges.

However, in case when $k > 0$ for some dimensions of cross-section, $\partial^2 H/\partial b^2 > 0$ may hold and then the Hamiltonian reaches its maximum only for $b = b_{min}$ or b_{max}, and the column has step-variable flanges with number of changes of width depending on the boundary conditions of the column. This property not only facilitates the calculations and but also allow to obtain the solution directly without any iterations, which is necessary in other cases.

Namely, the coordinates of changes of flange width and the critical load may be determined from the conditions of continuity of the Hamiltonian (21) and rotational displacement in the places of the cross-section change (see Appendix).

5. NUMERICAL EXAMPLES

The extreme critical loads and their corresponding shapes of flanges are determined for the simply supported I column with the following constraints imposed on the width of flanges

$$0.2 \leq b(z) \leq 0.4 \text{ m}, \tag{26}$$

and these results are presented in Fig. 2. The initial shape of flanges is shown in Fig. 2 by dashed lines.

The minimum critical load and the corresponding shape of flanges are obtained not only in iterative way but also on the base of the considerations given in Appendix and the latter result is shown in Fig. 3 by dashed line, whereas the maximum critical load and suitable shape of flanges are evaluated only in the iterative way.

In order to emphasize the difference between the flexural and the torsional buckling, the dependences of the extreme critical loads upon the volume of the flange for both considered kinds of buckling are sketched in Fig. 2. The history of optimum design vs number of iterations is presented in Fig. 3.

Moreover some differences have been observed in the post-buckling behavior of the I columns optimally shaped against torsional or flexural buckling which is shown in Fig. 4, where the initial post-buckling equilibrium paths are presented. In this figure Θ_0 and v_0 represent the value of the rotational displacement and the deflection in the middle of the column, respectively. The initial post-buckling equilibrium paths after the torsional buckling are determined by using the finite element procedure presented in [6]. However, the post-buckling path after the flexural buckling is drawn on the basis of Ref. [8]. The calculations are carried out for the ten-element discretization of a half of the column. Finally, a very good convergence of the described iterative procedure is worth noticing, only 3–4 iterations are necessary to obtain the presented results with on accuracy less than one per mille.

6. CONCLUSIONS

On the basis of the investigations carried out one can draw a conclusion that for the torsional buckling under conservative loads the critical load of a column with variable

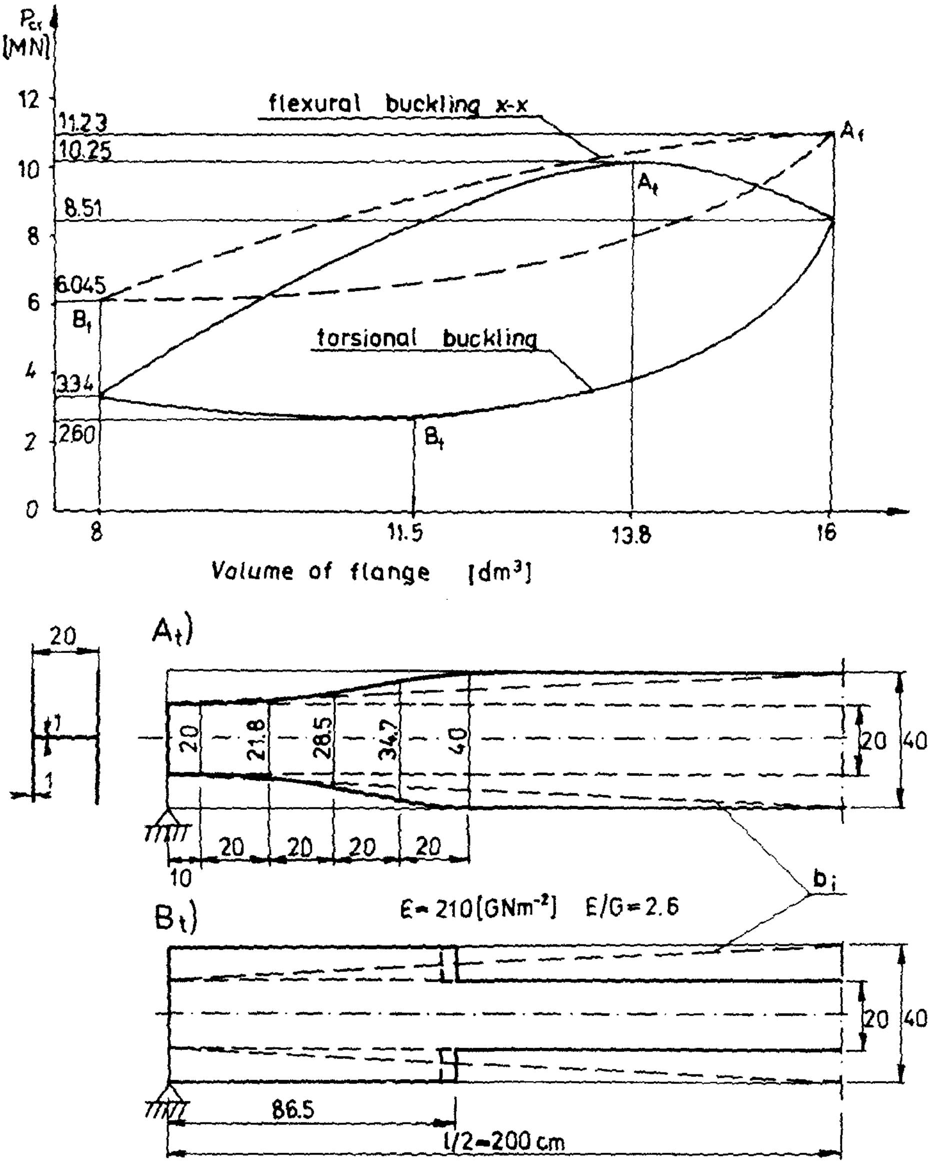

Fig. 2. Extreme critical loads vs volume of flange and optimal shapes of the I column flange.

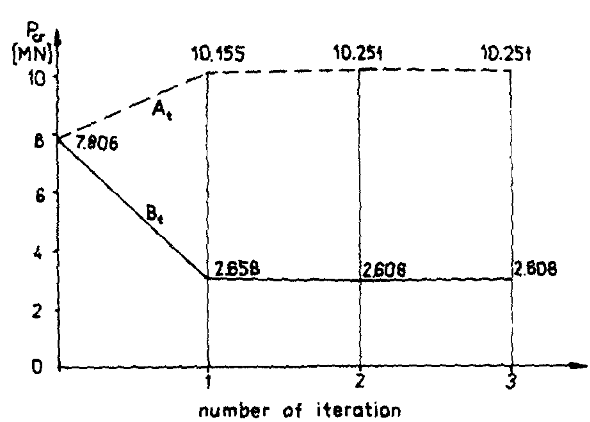

Fig. 3. History of optimum design of the I column vs number of iterations.

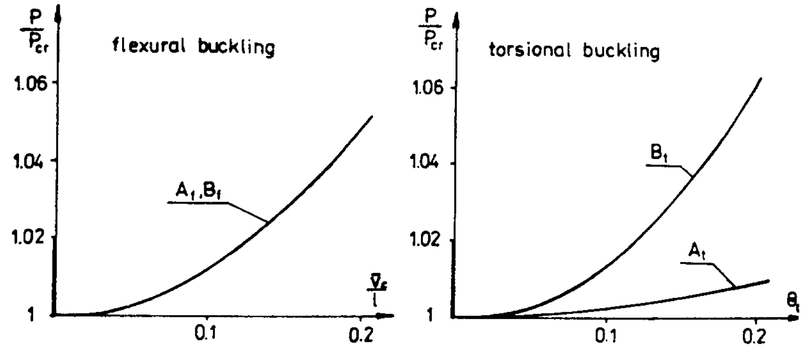

Fig. 4. Post-buckling equilibrium paths for I column.

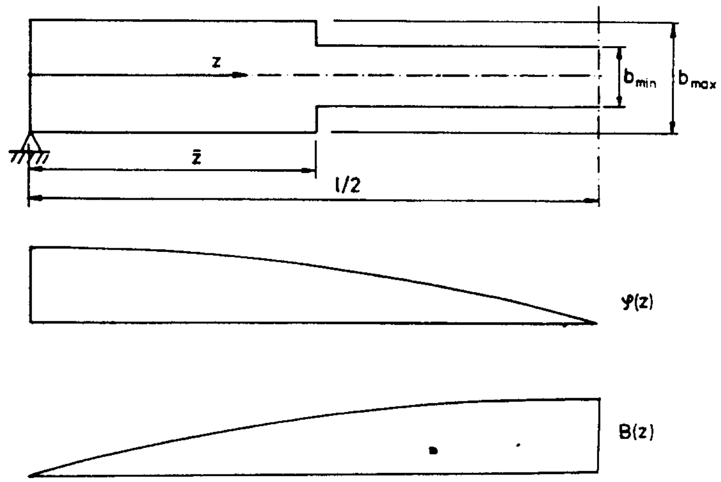

Fig. 5. Optimal shape of flange width for minimum critical load and plots of variables φ and B.

cross-section may exceed the bounds determined by critical loads for columns with extreme constant cross-section. It should also be noted that this effect is dependent on the dimensions of column and the boundary conditions, and in same cases it may be very distinct as in the example given in the paper but in some others it does not occur. Although the presented considerations deal only with the I column, the conclusion seems to be valid not only for any bisymmetric cross-section but also for other cases, however, the proof of the property, even in the case of I column with variable height of the web, would be very difficult with regard to serious mathematical difficulties of the derivation of fundamental differential equations. The property under consideration does not hold for the flexural buckling in which the extreme critical loads are achieved for the extreme constant cross-section (see Fig. 3).

Moreover, the post-buckling analysis allows us to state that for the optimally shaped column the point of bifurcation is symmetric and stable in both considered cases, which means that the critical load is insensitive to the unavoidable initial geometrical imperfections [8]. However, in the case of unstable or asymmetric points of bifurcation a considerable reduction of the critical load is possible so that reformulation of the optimization problem is necessary to include the effect of initial imperfections. Nevertheless for the flexural buckling of the optimally shaped column the post-buckling equilibrium path is independent of the dimensions of the cross-section, on the other hand, for the torsional buckling these post-buckling paths, determining the sensitivity of the point of bifurcation to the geometrical imperfection, are distinctly dependent on the dimensions of the cross-section.

Finally the good convergence of the presented numerical procedure in the determination of the optimal shape of the column should also be noted.

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APPENDIX

Let us consider a simply supported I column with step-variable width of flanges shown in Fig. 5. Here the functions $B(z)$ and $\varphi(z)$ corresponding to the critical load are also sketched. Taking into account the symmetry of the column, we will confine our attention to one half it only. The necessary condition of the Hamiltonian (21) maximum, after substitution (25), can be written as

$$\frac{\partial H}{\partial b} = k\left\{-\frac{72B^2}{Eh^2d_fb^4} + \left[P_{cr}\frac{1/3d_fd_wh^3 + 1/2d_fd_wb^2h + 2/3d_f^2b^3}{(d_wh + 2d_fb)^2} - \frac{2}{3}Gd_f^3\right]\varphi^2\right\} = 0. \tag{A1}$$

Taking into consideration the relation

$$B = -EI_\omega\varphi' = -E\frac{b^3h^2}{24}d_f\varphi', \tag{A2}$$

resulting from eqns (15) and (25) and substituting it into eqn (A1), it is evident that in the vicinity of the middle of the column where $\varphi \to 0$ it holds that $b \to 0$. However, in the neighborhood of the column ends $B \to 0$, then $b \to \infty$. On the basis of the above discussion, the supposed step variable shape of the flanges, satisfying the geometrical constraints (24), is presented in Fig. 5. So, we need two equations for calculation of the two unknown values, namely, the coordinate $\bar{z}$ of the step change of the cross-section, and the critical value of load P.

The first one can be obtained from the condition of continuity of the Hamiltonian in the place of change of the cross-section

$$\frac{B(\bar{z})^2}{EI_{\omega_1}} + \left(P_{cr}\frac{I_{01}}{A_1} - GI_{d_1}\right)\varphi(\bar{z})^2 = \frac{B(\bar{z})^2}{EI_{\omega_2}} + \left(P_{cr}\frac{I_{02}}{A_2} - GI_{d_2}\right)\varphi(\bar{z})^2, \tag{A3}$$

in which subscripts 1 and 2 refer to the values of the geometrical constants in the considered cross-section from the left and the right side, respectively. The values of functions $B(\bar{z})$ and $\varphi(\bar{z})$ are determined from the basic differential equation valid for the column with a constant cross-section

$$EI_\omega\Theta^{IV} + \left(P_{cr}\frac{I_0}{A} - GI_d\right)\Theta'' = 0. \tag{A4}$$

The general solution of the equation satisfying the boundary conditions

$$\begin{gathered}(1)\quad \Theta(0) = \Theta_0, \quad (2)\quad B(0) = -EI_\omega\Theta''(0) = B_0,\\ (3)\quad \Theta'(0) = \Theta_0', \quad (4)\quad M(0) = \left(GI_d - P_{cr}\frac{I_0}{A}\right)\Theta'(0) - EI_\omega\Theta'''(0) = M_0,\end{gathered} \tag{A5}$$

has the form

$$\Theta(z) = \Theta_0 + \Theta_0'\frac{\sin\lambda z}{\lambda} + B_0\frac{1}{\lambda^2EI_\omega}(\cos\lambda z - 1) + M_0\frac{1}{\lambda^3EI_\omega}\left(\frac{\sin\lambda z}{\lambda} - z\right), \tag{A6}$$

where the description $\lambda = \sqrt{((P_{cr}(I_0/A) - GI_d)/EI_\omega)}$ is introduced. In the first part of the column $(0 \le z \le \bar{z})$ due to the boundary conditions: $1/\Theta(0) = 0, 2/B(0) = 0$, and $3/M(0) = 0$ which results from the property $M(z) = \text{constant}$ and the boundary condition in the middle of the column $M(l/2) = 0$, we have the following solution

$$\Theta(z) = \Theta_0'\frac{\sin\lambda_1 z}{\lambda_1}. \tag{A7}$$

Now, we can determine

$$\begin{gathered}B(\bar{z}) = EI_{\omega 1}\Theta_0'\lambda_1\sin\lambda_1\bar{z},\\ \varphi(\bar{z}) = \Theta'(\bar{z}) = \Theta_0'\cos\lambda_1\bar{z}.\end{gathered} \tag{A8}$$

Substituting the relations (A8) into (A3) after some algebra we obtain the first equation

$$\tan^2 \lambda_1 \bar{z} - \frac{P_{cr}\left(\frac{I_{02}}{A_2} - \frac{I_{01}}{A_1}\right) - G(I_{d2} - I_{d1})}{EI_{\omega_1}\lambda_1^2\left(1 - \frac{I_{\omega_1}}{I_{\omega_2}}\right)} = 0. \tag{A9}$$

The second one will be found from the boundary condition in the middle of the column

$$\Theta'\left(z = \frac{l}{2}\right) = 0. \tag{A10}$$

To utilize this condition we must, at first, determine the solution of the eqn (A4) valid in the second part of the column

$$\Theta(z) = \Theta(\bar{z}) + \varphi(\bar{z})\frac{\sin \lambda_2(z - \bar{z})}{\lambda_2} + \frac{B(\bar{z})}{\lambda_2^2 EI_{\omega_2}}[\cos \lambda_2(z - \bar{z}) - 1]. \tag{A11}$$

Making use of relations (A8), (A10), and (A11) after some calculations, we arrive at the finished form of the second equation

$$\tan \lambda_2\left(\frac{l}{2} - \bar{z}\right)\tan \lambda_1\bar{z} + \frac{GI_{d_2} - P_{cr}\frac{I_{02}}{A_2}}{EI_{\omega_1}\lambda_1\lambda_2} = 0. \tag{A12}$$

The eqns (A9) and (A12) allow us to determine two sought values: $\bar{z}$ and P_{cr}.